$Y(x)$ on x is no hindrance in the method proposed.

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ON THE STUDY OF MINIMAX EVALUATION OF PARAMETERS OF NON-LINEAR SYSTEMS *

V.G. POKOTILO

The problem of the a posteriori minimax evaluation /1/ of the unknown parameters of non-linear systems of fairly general form is investigated. Approximations of information sets related to the observation process are defined using the non-linear theory of duality. The asymptotic properties of minimax estimates are also obtained in the case of perturbations that can be represented in the form of random processes. Problems of minimax observation as applied to non-linear systems were investigated in /2, 3/.

1. Let us assume that the observed signal conforms to the equation

$$y(t) = g(t, z, w(t)), \quad t \in [0, T] \tag{1.1}$$

where the unknown vector of the parameters $z \in R^n$ and perturbation $w(T; \cdot) = \{w(t), t \in [0, T]\}$ satisfy constraints of the form

$$z \in Z^\circ, \quad w(t) \in W(t) \subseteq W, \quad t \in [0, T]$$

The m-vector of the function $g(\cdot, \cdot, \cdot)$ and the input data are assumed to be as follows

1) Z° and W are compact in R^n and R^s, respectively;

2) $g(t, z, w)$ is continuous with respect to the set of variables and, moreover, the set of functions $\{g(t, z, \cdot), t \geqslant 0\}$ is equicontinuous for any $z \in Z^\circ$;

3) the class of admissible perturbations is defined by the set

$$E = \{w(T; \cdot) \in C^s[0, T]: w(t) \in W(t), \; t \in [0, T]\}$$

4) the set of possible outputs of system (1.1)

$$G(z) = \{f(T; \cdot): f(t) = g(t, z, w(t)); \quad w(T; \cdot) \in E\} \tag{1.2}$$

is closed in space $C^m[0, T]$.

Definition 1.1. The set

$$Z(T; y(\cdot)) = \{z': y(T; \cdot) \in G(z')\}$$

is called the information set, compatible with the signal $y(T; \cdot) = \{y(t), t \in [0, T]\}$.

Points $z_*(T) \in Z(T; y(\cdot))$, somehow separated, will be called the a posteriori minimax evaluations of the vector of parameters z.

To describe the weak dependence of random processes, which simulate perturbations in a stochastic system, we use following definition.

Definition 1.2. The random process $\{w(t), t \geqslant 0\}$ in the probability space $\{\Omega, \Sigma, P\}$ with the phase space $\{R^s, \Delta\}$ is called entirely regular, if

$$\alpha(\tau) = \sup_{A \in \Gamma_0^t,\, B \in \Gamma_{t+\tau}^\infty,\, t \geqslant 0} |P(AB) - P(A)P(B)| \to 0$$

as $\tau \to \infty$, where $\Gamma^b{}_a$, $0 \leqslant a \leqslant b \leqslant +\infty$, is the σ-algebra generated by $\{w(t), a \leqslant t \leqslant b\}$.

We will present without proof the statement that defines the entirely regular processes.

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Lemma 1.1. Let $\{w(t),\ t \geqslant 0\}$ be an entirely regular random process and $A_i \in \Gamma_{t_{i-1}}^{t_i},\ t_i > t_{i-1}$, such that

$$P(A_i) < 1 - \varepsilon,\quad \varepsilon > 0,\quad i = 1, 2, \ldots;\quad t_i \to \infty$$

as $i \to \infty$.

Then

$$P(\bigcap_{i=1}^{N} A_i) \to 0,\quad N \to \infty$$

2. Let A and B be closed sets in R^n. Denoting by $\delta(A, B)$ the deviation of set A from set B /5/, we obtain

$$\delta(A, B) = \inf\{\varepsilon \geqslant 0\colon A \subset B + \varepsilon S\}$$

Here and henceforth S is the unit sphere in the corresponding space.

Theorem 2.1. Let $\{w(t),\ t \geqslant 0\}$ be an entirely regular random process whose realizations P are almost certainly admissible perturbations, and for any $t \geqslant 0$, $\varepsilon > 0$ and $w_* \in W(t)$

$$P\{w(t) \in (w_* + \varepsilon S) \cap W(t)\} \geqslant \pi(\varepsilon) > 0 \tag{2.1}$$

Then with unit probability

$$\lim_{T\to\infty} \delta(Z(T; y(\cdot)), I(z)) = 0$$
$$I(z) = \{z' \in Z^\circ\colon \sup_{\gamma>0} \lim_{t\to\infty} \delta(g(t, z, W(t)), g(t, z' + \gamma S, W(t))) = 0\}$$

Proof. We set the sequences $\{t_s,\ s = 1, 2, \ldots\}$ and $\{w_s \in W(t_s), s = 1, 2, \ldots\}$ in correspondence with point $z_* \notin I(z)$, and the positive numbers $\gamma(z_*)$, $\varepsilon(z_*)$, such that

$$g(t_s, z, w_s) \notin g(t_s,\ z_* + \gamma(z_*) S, W(t_s)) + \varepsilon(z_*) S$$

The function $\gamma(\cdot)$ may be selected to be semicontinuous from below at the point z_*.

Indeed, for some sequences $z_k \to z_*$ let there be a $\nu_* > 0$, such that for all reasonably large k a $\gamma(z_k) < \gamma(z_*) - \nu_*$ exists. It can be assumed here, without loss of generality, that $\varepsilon(z_k) \leqslant \varepsilon(z_*)$, hence for any t_s and $w_s \in W(t_s)$, $s = 1, 2, \ldots$ we have

$$g(t_s, z, w_s) \in g(t_s, z_k + (\gamma(z_*) - \nu_*) S,\ W(t_s)) + \varepsilon(z_*) S;\ s, k = 1, 2,$$

Consequently, for reasonably large k

$$g(t_s, z, w_s) \in g(t_s, z_* + \gamma(z_*) S, W(t_s)) + \varepsilon(z_*) S$$

which contradicts the definition of $\gamma(z_*)$ and $\varepsilon(z_*)$.

We fix the arbitrary $\delta > 0$ and consider the compact set $K_\delta = Z^\circ \setminus (I(z) + \delta S)$. Let $\gamma = \min\{\gamma(z),\ z \in K_\delta\}$ and points z_i, $i = 1, 2, \ldots, N$ form the γ-net of the set K_δ. Then

$$\{\delta(Z(T; y(\cdot)), I(z)) > \delta\} = \{Z(T; y(\cdot)) \cap K_\delta \neq \varnothing\} \subset \bigcup_{i=1}^{N} \{Z(T; y(\cdot)) \cap (z_i + \gamma S) \neq \varnothing\} \tag{2.2}$$

Let us determine for each $i = 1, 2, \ldots, N$ the sequences t_s^i, $w_s^i \in W(t_s^i)$, $s = 1, 2, \ldots;$ as $t_s^i \to \infty$, $s \to \infty$, and the numbers ε_i such that

$$g(t_s^i, z, w_s^i) \notin g(t_s^i, z_i + \gamma S, W(t_s^i)) + \varepsilon_i S;\quad s = 1, 2, \ldots$$

With the assumption made regarding the function $g(\cdot, \cdot, \cdot)$, we have from the last relation

$$g(t_s^i, z, w_s^i + \delta_i S) \notin g(t_s^i, z_i + \gamma S, W(t_s^i));\quad s = 1, 2, \ldots$$

for certain $\delta_i > 0$, $i = 1, 2, \ldots, N$.

Consequently,

$$\{Z(T; y(\cdot)) \cap (z_i + \gamma S) \neq \varnothing\} \subset \bigcap_{s=1}^{R_i(T)} \{w(t_s^i) \notin w_s^i + \delta_i S\}$$
$$R_i(T) = \max\{s : t_s^i \leqslant T\}$$

and by virtue of (2.2) the following inclusion holds:

$$\{\delta(Z(T; y(\cdot)), I(z)) > \delta\} \subset \bigcup_{i=1}^{N} \bigcap_{s=1}^{R_i(T)} \{w(t_s^i) \notin w_s^i + \delta_i S\} \tag{2.3}$$

Using the condition of transfer of the measure P in the space of continuous functions with Borel σ-algebra /6/, it can be shown that under the conditions considered here

$$\{\delta(Z(T; y(\cdot)), I(z)) > \delta\} \in \Sigma$$

Thus from (2.3) we obtain the following inequality:

$$P\{\delta(Z(T;y(\cdot)), I(z)) > \delta\} \leqslant \sum_{i=1}^{N} P(\bigcap_{s=1}^{R_i(T)} \{w(t_s^i) \notin w_s^i + \delta_i S\})$$

Since $R_i(T) \to \infty$ as $T \to \infty$, $i = 1, 2, \ldots, N$, it follows from (2.1) and Lemma (1.1) that for any $i = 1, 2, \ldots, N$

$$P\Big(\bigcap_{s=1}^{R_i(T)} \{w(t_s^i) \notin w_s^i + \delta_i S\}\Big) \to 0, \quad T \to \infty$$

Hence

$$P\{\delta(Z(T; y(\cdot)), I(z)) > \delta\} \to 0, \quad T \to \infty$$

i.e. $\delta(T) = \delta(Z(T; y(\cdot)), I(z)) \to 0$ according to the probability P. The convergence with unit probability follows simply from the monotonicity of $\delta(T)$ in T. The theorem is proved.

Corollary 2.1. If the parameters from Z° in signal (1.1) are distinguishable in the sense that $I(z) = \{z\}$ for any $z \in Z^\circ$, the minimax evaluations $z_*(T)$ are strictly justifiable.

Note 2.1. If the signal (1.1) is formed at the output of a linear system $g(t, z, w) = \theta(t)z + w$ and $\|\theta(t)\|$ is bounded when $t \geqslant 0$, then

$$I(z) = \{z' \in Z^\circ : \lim_{t\to\infty} \|\theta(t)(z - z')\| = 0\}$$

In that case the condition $I(z) = \{z\}$ may be represented in the form

$$\lim_{t\to\infty} \|\theta(t)z\| > 0, \quad \forall z \in R^n, \quad z \neq 0$$

3. To define the approximations of information sets we use the methods of the non-linear theory of duality. Consider, as in /7, 8/, the following generalization of the concepts of conjugate function and the subdifferential in convex analysis.

Let $\varphi(\cdot,\cdot): R^p \times R^n \to \bar{R} = R \cup \{-\infty, +\infty\}$; $f(\cdot)$: $R^n \to \bar{R}$ and $g(\cdot)$: $R^p \to \bar{R}$ be arbitrary functions.

Definition 3.1. /8/. We call the functions

$$f^{\cap}(\psi) = \inf\{f(x) + \varphi(\psi, x),\ x \in R^n\}$$
$$g^{\cup}(x) = \sup\{g(\psi) - \varphi(\psi, x),\ \psi \in R^p\}$$

the lower and upper φ-conjugate functions of the functions $f(\cdot)$ and $g(\cdot)$, respectively.

Definition 3.2. The vector $\psi \in R^p$ is called the φ-subdifferential of the function $f(\cdot)$ at the point x_0, if

$$f(x) + \varphi(\psi, x) \geqslant f(x_0) + \varphi(\psi, x_0), \quad \forall x \in R^n$$

The set of subdifferentials of the function $f(\cdot)$ is denoted at the point x_0 by $\partial_\varphi f(x_0)$.

Subsequently we assume that the function $\varphi(\psi, x)$ is continuous in x. Simultaneously with the support function $\rho(\cdot \mid X)$ of set X we shall consider also the φ-support function of this set which is defined by the equation

$$\rho_\varphi(\psi \mid X) = \inf\{\varphi(\psi, x),\ x \in X\}$$

Assuming that the sets $Z(T; y(\cdot))$ are approximated by measuring the signal (1.1) only at a finite number of points, we consider the set of piecewise constant functions in the interval $[0, T]$.

Theorem 3.1. The inclusion

$$Z(T; y(\cdot)) \subset \{z' : \varphi(\psi, z') \geqslant \gamma(T; \psi), \quad \forall \psi \in R^p\}$$
$$\gamma(T; \psi) = \sup\{F^{\cap}(\lambda(\cdot); \psi) - \langle\lambda(\cdot), y(\cdot)\rangle;\ \lambda(\cdot) \in \Pi\}$$
$$F(\lambda(\cdot); z) = \rho(\lambda(\cdot) \mid G(z));\ \langle\lambda(\cdot), y(\cdot)\rangle = \int_0^T d\lambda(t)\, y(t)$$

holds.

Proof. By definition of the information set, $z \in Z(T; y(\cdot))$, if and only if

$$\langle\lambda(\cdot), y(\cdot)\rangle - \langle\lambda(\cdot), f(\cdot)\rangle \geqslant 0, \quad \forall \lambda(\cdot) \in V^m[0, T] \tag{3.1}$$

for some possible output of system (1.1). For the system of inequalities (3.1) to be compatible on the set $G(z)$ defined by relation (1.2), it is necessary that

$$\langle\lambda(\cdot), y(\cdot)\rangle - F(\lambda(\cdot); z) \geqslant 0, \quad \forall \lambda(\cdot) \in V^m[0, T]$$

Using the analogue of the Young-Fenchel inequality, that follows from definition 3.1, we obtain that for all $\lambda(\cdot) \in V^m[0, T]$

$$\langle \lambda(\cdot),\ y(\cdot)\rangle - F^{\cap}(\lambda(\cdot);\psi) \geqslant -\varphi(\psi, z)$$

and, consequently,

$$\varphi(\psi, z) \geqslant \sup\{F^{\cap}(\lambda(\cdot);\psi) - \langle \lambda(\cdot), y(\cdot)\rangle;\ \lambda(\cdot) \in \Pi\}$$

The theorem is proved.

To obtain a definition of the approximations of the sets $Z(T; y(\cdot))$ in terms of their support functions we shall cite several statements without proof.

Lemma 3.1. The positive homogeneity of the function $\gamma(T;\cdot)$ follows from the positive homogeneity of the function $\varphi(\cdot, x)$. If, in addition, $\varphi(\cdot, x)$ is a concave function, then $\gamma(T;\cdot)$ is a positive homogeneous concave function.

Lemma 3.2. If $\varphi(\cdot, x)$ and $\omega(\cdot)$ as $R^p \to \bar{R}$ are positive-homogeneous functions and $\omega(\psi) = \omega^{\cup\cap}(\psi)$ for all $\psi \in R^p$, then $\omega(\cdot)$ is a φ-supporting function of some set.

From theorem (3.1) and Lemmas 3.1 and 3.2 we obtain the following corollary.

Corollary 3.1. If $\varphi(\cdot, x)$ is a positive-homogeneous function then

$$Z(T; y(\cdot)) \subset Z_1(T; y(\cdot)) \subset Z^{\circ}$$
$$(\rho_{\varphi}(\psi \mid Z_1(T; y(\cdot))) = \gamma^{\cup\cap}(T;\psi))$$

It also follows from Lemma 3.1 that in certain cases it is possible to define the approximations of information sets by using the closing of the function $\gamma(T;\cdot)$ in the sense of convex analysis. In particular the following corollary holds.

Corollary 3.2. If $\varphi(\psi, z) = \langle \psi, z\rangle + \langle \psi_1, f(z)\rangle$, then

$$Z(T; y(\cdot)) \subset Z_2(T; y(\cdot)) = \{z':\ (-z',\ -f(z')) \in Z_3(T; y(\cdot))\}$$
$$\rho(\psi \mid Z_3(T; y(\cdot))) = \chi^{**}(\psi);\quad \chi(\psi) = -\gamma(T;\psi)$$

Here $\chi^{**}(\cdot)$ is the second conjugate function in the usual sense of convex analysis (see, e.g, /5/).

Note 3.1. For linear systems when $g(t, z, w) = \theta(t) z + w$, setting $\varphi(\psi, z) = -\langle \psi, z\rangle$, we obtain $f^{\cap}(\psi) = -f^{*}(\psi)$ and

$$\gamma(T;\psi) = -\inf\left\{\rho(-\lambda(\cdot)\mid E) + \langle \lambda(\cdot),\ y(\cdot)\rangle : \psi = \int_0^T d\lambda(t)\,\theta(t),\ \lambda(\cdot) \in \Pi\right\}$$

Hence, if the sets Z^0 and $W(t)$ are convex, and the multivalued mapping $W(\cdot)$ is continuous in the Hausdorff metric, the equation

$$\rho(\psi \mid Z(T; y(\cdot))) = -\gamma(T;\psi)$$

holds /1/.

4. We will now study the asymptotic properties of the approximations of information sets obtained in Sect.3.

We define for $L, T \geqslant 0$ and natural l the sets

$$\Theta(l, L, T) = \{t_k,\ k = 1, 2, \ldots, l;\ t_1 \geqslant T;\ t_{k+1} \geqslant t_k + L,\ k = 1, 2, \ldots, l\}$$
$$\Lambda(l, L, T) = \Big\{\lambda(\cdot) \in \Pi : \frac{d\lambda(t)}{dt} = \sum_{k=1}^{l} \lambda_k \delta(t - t_k),\ \|\lambda_k\| \leqslant 1,$$
$$\{t_k, k = 1, 2, \ldots, l\} \in \Theta(l, L, T)\Big\};$$
$$\Psi(z) = \bigcup_{l=1}^{\infty} \operatorname{con}\Big(\bigcap_{L, T \geqslant 0} \{\partial_{\varphi} F(\lambda(\cdot); z);$$
$$\lambda(\cdot) \in \Lambda(l, L, T)\}\Big);\quad \Psi = \bigcap_{z' \in Z^{\circ}} \Psi(z')$$

Here $\operatorname{con} X$ denotes the conical envelope of the set X and $\partial_{\varphi} F(\lambda(\cdot); z)$ is the φ-subdifferential of the function $F(\lambda(\cdot);\cdot)$ at the point z.

Theorem 4.1. Under the conditions of Theorem 2.1

$$\rho_{\varphi}(\psi \mid Z_1(T; y(\cdot))) \to \varphi(\psi, z) \quad \text{as} \quad T \to \infty$$

with unit probability for all $\psi \in \Psi$.

Proof. Using the definition of the φ-subdifferential and the analogue of the Young-Fenchel inequality, we can show that

$$\gamma(T;\psi) \geqslant \varphi(\psi, z) + \omega(T;\psi, z) \tag{4.1}$$

$$\omega(T;\psi,z)=\sup\{F(\lambda(\cdot);z)-\langle\lambda(\cdot),y(\cdot)\rangle:\psi\in\partial_\varphi F(\lambda(\cdot);z);\ \lambda\in\Pi\}$$

We fix $\psi\in\Psi\subset\Psi(z)$. By the definition of the cone $\Psi(z)$ and $M\geqslant 0$ and a natural l exist such that for any T and L some $\{t_k, k=1,2,\ldots,l\}\in\Theta(l,L,T)$ and $\lambda(\cdot)\in\Pi$ can be found for which the inclusion

$$M^{-1}\lambda(\cdot)\in\Lambda(l,L,T),\quad \psi\in\partial_\varphi F(\lambda(\cdot);z)$$

holds.

Consequently it is possible to indicate for each L a set of pairs $\{(\lambda_{ik}, t_{ik}), k=1,2,\ldots, l, i=1,2,\ldots\}$ such that

$$\{t_{ik}, k=1,2,\ldots,l\}\in\Theta(l,L,t_{i1}),\quad t_{i1}\to\infty,\quad i\to\infty$$

$$\|\lambda_{ik}\|\leqslant M;\quad \psi\in\partial_\varphi F(\lambda_i(\cdot);z);\quad \frac{d\lambda_i(t)}{dt}=\sum_{k=1}^{l}\lambda_{ik}\delta(t-t_{ik})$$

We put

$$A_i=\{\sum_{k=1}^{l}[\rho(-\lambda_{ik}\,|\,g(t_{ik},z,W(t_{ik}))+\lambda_{ik}g(t_{ik},z,w(t_{ik}))]\geqslant\varepsilon\}$$

By virtue of the regularity of the process $\{w(t), t\geqslant 0\}$ and condition 2.1 the inequality $P(A_i)<1-\delta$ holds for some $\delta>0$ and fairly large L.

Having thus set the number L and the set $\{(\lambda_{ik}, t_{ik}), k=1,2,\ldots,l, i=1,2,\ldots\}$ related to it, from (4.1) we obtain

$$\begin{aligned}&\{|\gamma(T;\psi)-\varphi(\psi,z)|\geqslant\varepsilon\}=\{\varphi(\psi,z)-\gamma(T;\psi)\geqslant\varepsilon\}\subset\\&\{-\omega(T;\psi,z)\geqslant\varepsilon\}\subset\bigcap_{i=1}^{N(T)}A_i,\\&N(T)=\max\{N:t_{il}\leqslant T,\ i=1,2,\ldots,N\}\end{aligned}\qquad(4.2)$$

It can be shown that under the conditions here $\{\varphi(\psi,z)-\gamma(T;\psi)\geqslant\varepsilon\}\in\Sigma$, and hence the theorem follows from (4.2). Lemma 1.1, and the monotonicity of $\gamma(T;\psi)$ with respect to T.

By virtue of Note 3.1 the theorem proved here directly generalizes the basic results of /9, 10/.

5. When using the usual concepts of duality to the analysis of non-linear systems, based on the notation of conjugate functions in convex analysis, the degeneracy of the functions $\gamma(T;\psi)$ and $\omega(T;\psi,z)$ would be a characteristic feature. These functions are defined by relations (3.1) and (4.1) for all $\psi\in R^p$ and $\partial_\varphi F(\lambda(\cdot);z)=\varnothing$ for all $\lambda(\cdot)\in V^m[0,T])$ and, as a corollary, we would have degeneracy in stochastic systems of the conditions of convergence of the approximations of informations sets. We shall show, using a simple example, that the use of non-linear constructions of duality enables us to avoid this.

Example 5.1. Let

$$y(t)=\exp(zt)+w(t),\ w(t)\in[-1,1],\ Z^0=[0,a],\ a>0$$

Then

$$F(\lambda(\cdot);z)=\int^{T}\exp(zt)\,d\lambda(t)+\operatorname{Var}\lambda(\cdot)$$

and for $\lambda_t(\cdot):\dfrac{d\lambda_t(s)}{ds}=\lambda_t\delta(s-t)$ и $\varphi(\psi,z)=\psi z+\psi_1z^2,\ \lambda_t\geqslant 0$ we obtain

$$\partial_\varphi F(\lambda_t(\cdot);z)=\{(-\psi,\psi_1):\lambda_t t\exp(zt)+2\psi_1 z+\psi=0,\ \psi_1\geqslant 0\}$$

It can be seen that when $z\geqslant 0$

$$\{\partial_\varphi F(\lambda(\cdot);z);\lambda(\cdot)\in\Lambda(1,L,T)\}\supset\{\partial_\varphi F(\lambda(\cdot);z);\lambda(\cdot)\in\Lambda(1,0,0)\}$$

and consequently $\Psi\neq\varnothing$. In particular

$$\{(\psi,\psi_1):\psi\leqslant 0,\ \psi_1=0\}\subset\Psi,\ \forall a>0$$

Example 5.2. Consider the signal

$$y(t)=z(1+w(t)),\ w(t)\in[-1,1]$$

Here $F(\lambda_t(\cdot);z)=\lambda_t z-|\lambda_t|\,|z|$, and unlike the previous case, the function $F(\lambda(\cdot);\cdot)$ is not convex. Because of this the usual duality constructions prove to be unsuccessful since $F^*(\lambda(\cdot);\psi)=-\infty$, $\forall\lambda(\cdot)\in V[0,T]$, and $\psi\in R$, while, at the same time, selecting $\varphi(\psi,z)=\psi z+\psi_1|z|$, we obtain

$$\Psi(z)\supset\operatorname{con}\{\partial_\varphi F(\lambda_t(\cdot);z);\ |\lambda_t|\leqslant 1\}$$

$$\Psi\supset\{(\psi,\psi_1);\ |\psi|\leqslant\psi_1\}$$

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THE THEORY OF DUALITY IN SYSTEMS WITH AFTEREFFECT*

B.SH. MORDUKHOVICH

Some problems of control and observation /1—3/ of linear dynamic systems with aftereffect, defined by differential and integral equations with deviating arguments are considered. The theory of duality for the problem of minimizing the Boltz convex functional on the trajectories of a functionally differentiable system of the neutral type with a lag in the control, state, and velocity variables is developed. New concepts of controlliability are introduced into the system with aftereffects and phase constraints, as well as dual concepts of ideal observability of their conjugate system of integral equations with a lead in conditions of incomplete information. The observability concepts introduced here are connected with the restitution of the generalized final state of the system containing minimum information to enable the future motion to be calculated uniquely. The schemes and results obtained enable them to be used in differential-game problems of dynamic systems with aftereffects /4—6/.

1. The problem of optimal control. Consider a linear control system whose dynamics along the segment $[t_0, t_1]$ is defined by differential equations with a deflecting argument of the neutral type

$$x^{\cdot}(t) = A(t)x(t) + A_1(t)x(t-h) + A_2(t)x^{\cdot}(t-h) + B(t)u(t) + B_1(t)u(t-h) \tag{1.1}$$

where $h > 0$ is the lag of the control, state and velocity variables.

Systems with an aftereffect of the type (1.1) occur in problems of mechanics, automatic control, economics, etc. (see the numerous examples in /7/). It is important to allow for the action of the aftereffect when defining real dynamic systems and related control and observation processes.

Let us consider the problem of minimizing the Boltz functional

$$I(x, u) = \Phi(x(t_0), x(t_1)) + \int_{t_0}^{t_1} F(x(t), u(t), t)\, dt \to \inf \tag{1.2}$$

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